Network right $*$-abundant semigroups

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York Seminar

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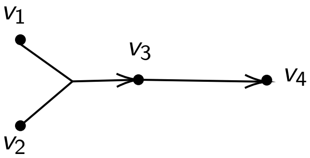
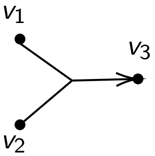
Preliminaries: networks

Definition 1 A **network** $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ consists of the set of vertices V, the set of relations T, together with mappings $\mathbf{s}, \mathbf{r} : T \to \mathcal{P}(V)$, respectively called the *source mapping* and the *range mapping* for Γ, where $\mathcal{P}(V)$ is the power set of V and for all $t \in T$, $\mathbf{s}(t)$ and $\mathbf{r}(t)$ are disjoint non-empty subsets of V.



t_1 :



$s(t_1) = \{v_1,\ v_2\}$

$r(t_1) = \{v_3\}$

$V = \{v_1,\ v_2,\ v_3,\ v_4\}$

$T = \{t_1, t_2\}$

t_2 :

$s(t_2) = \{v_3\}$

$r(t_2) = \{v_4\}$

Preliminaries: networks

Remark: If for all $t \in T$ $\mathbf{s}(t)$ and $\mathbf{r}(t)$ are singletons in a network $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ we identify Γ with the underlying simple direct graph and refer to it simply as a **graph**.

Let $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ be a network. Put

$$T^0 = \{A \subseteq V : \exists\ t \in T, A = \mathbf{s}(t) \text{ or } A = \mathbf{r}(t)\} \cup V,$$

and for all $A \in T^0$, $\mathbf{s}(A) = \mathbf{r}(A) = A$.

Definition 2 A **path** in a network $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ is a finite sequence $\alpha = t_1 t_2 \cdots t_n$ of elements of $T \cup T^0$ such that $\mathbf{r}(t_i) \cap \mathbf{s}(t_{i+1}) \neq \emptyset$ for $i = 1, 2, \cdots, n-1$. In such a case, $\mathbf{s}(\alpha) = \mathbf{s}(t_1)$ is the *source* of α, $\mathbf{r}(\alpha) = \mathbf{r}(t_n)$ is the *range* of α.

An element in T^0 is said to be an **empty path**.

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An element in T^0 is said to be an **empty path**.

Preliminaries: networks

Let $P(\Gamma)$ denote the set of all paths in a network Γ. This includes the zero-element 0 and T^0.

$$P(\Gamma) = \{t_1 \cdots t_n : t_i \in T \cup T^0, \mathbf{r}(t_i) \cap \mathbf{s}(t_{i+1}) \neq \emptyset \text{ for } i = 1, \cdots, n-1\} \cup \{0\}$$

Definition 3 If $\mathbf{r}(t_i) = \mathbf{s}(t_{i+1})$ for $i = 1, \cdots, n-1$ in $\alpha = t_1 \cdots t_n \in P(\Gamma)$, then α is said to be a **linear path**.

Let $LP(\Gamma)$ denote the set of all linear paths in Γ. This includes T^0 but does not contain the zero element 0.

$$LP(\Gamma) = \{t_1 \cdots t_n \in P(\Gamma) \setminus \{0\} : \mathbf{r}(t_i) = \mathbf{s}(t_{i+1}) \text{ for } i = 1, \cdots, n-1\}.$$

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1. 1962, Leavitt path algebra —Leavitt, W. G.
2. 1975, Graph inverse semigroups —Ash, C. J. and Hall, T. E.
3. 1998, Cuntz-Krieger graph C^*-algebras —Kumjian, A., Pask, D. and Raeburn, I.
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Questions There exists a kind of paths $p = t_1 t_2 \cdots t_n$ of relations $t_i \in T_\Gamma$ with either $\mathbf{r}(t_i) = \mathbf{s}(t_{i+1})$ or $\mathbf{r}(t_i) \neq \mathbf{s}(t_{i+1})$ but $\mathbf{r}(t_i) \cap \mathbf{s}(t_{i+1}) \neq \emptyset$ for $i = 1, \ldots, n-1$ in a network. Naturally, one may ask what kind of algebraic systems such paths may lead to?

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Preliminaries: right $*$ abundant semigroups

- Green's star relations
- Right $*$-abundant semigroups
- Right ample semigroups

Green's star relations

Let S be a semigroup. For all $a, b \in S$,

$$a\mathcal{L}^* b \Leftrightarrow \forall x, y \in S^1 \big(ax = ay \Leftrightarrow bx = by \big)$$

and

$$a\mathcal{R}^* b \Leftrightarrow \forall x, y \in S^1 \big(xa = ya \Leftrightarrow xb = yb \big).$$

Notice that $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$ on any semigroup S. For any regular elements $a, b \in S$, $(a, b) \in \mathcal{L}^*$ if and only if $(a, b) \in \mathcal{L}$ and $(a, b) \in \mathcal{R}^*$ if and only if $(a, b) \in \mathcal{R}$. In particular, if S is a regular semigroup, then $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$.

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Right $*$-abundant semigroups

A semigroup is said to be a **right abundant** semigroup if each $\mathcal{L}^*$-class contains an idempotent.

Definition 4 A right abundant semigroup S is said to be a **right $*$-abundant** semigroup if each $\mathcal{L}^*$-class of S contains a unique idempotent.

Let S be a right $*$-abundant semigroup with set of idempotents $E(S)$. We denote the unique idempotent of $E(S)$ in the $\mathcal{L}^*$-class of a by a^*.

Then * is a unary operation on a right $*$-abundant semigroup S and we may regard S as an algebra of type $(2, 1)$ and call such algebras **unary algebras**; as such, morphisms must preserve the unary operation of * (and hence the relation $\mathcal{L}^*$). Of course, any semigroup isomorphism must preserve the additional operations.

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Right ample semigroups

Definition 5 A **right ample** semigroup is defined to be a semigroup such that

(i) each $\mathcal{L}^*$-class contains an idempotent;

(ii) the idempotents commute;

(iii) $ea = a(ea)^*$ for any element a in S and any idempotent e in S.

Dually, a **left ample** semigroup is defined. An **ample** semigroup is defined to be a left and right ample semigroup.

In particular, an inverse semigroup is ample, where $a^\dagger = aa^{-1}$ and $a^* = a^{-1}a$, where $a^\dagger$ is the unique idempotent in the $\mathcal{R}^*$-class containing a and a^{-1} is the inverse of a.

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Constructions

- Graphs $\Rightarrow$ Graph inverse semigroups
- Networks $\Rightarrow$ Right $*$-abundant semigroups
- Right ample semigroups
- Inverse semigroups

Graphs ⇒ Graph inverse semigroups

Let $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ be a graph. For any $t \in T \cup V$, we define t^{-1} to be a relation with

$$\mathbf{s}(t^{-1}) = \mathbf{r}(t) \text{ and } \mathbf{r}(t^{-1}) = \mathbf{s}(t).$$

Notice that $v^{-1} = v$ for each $v \in V$. Put

$$T^{-1} = \{t^{-1} : t \in T\}.$$

Definition 6 Let $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ be a graph. The **graph inverse semigroup** is given by the presentation $I_\Gamma := \langle X : R \rangle$ where

$$X = T \cup V \cup T^{-1} \cup \{0\}$$

and R consists of the following relations:
(V) $uv = \delta_{uv} u$ for all $u, v \in V$;
(E) $\mathbf{s}(t)t = t = t\mathbf{r}(t)$ for each $t \in T \cup T^{-1}$;
(CK1) $t_1^{-1} t_2 = \delta_{t_1 t_2} \mathbf{r}(t_1)$ for all $t_1, t_2 \in T$;
(O) $0x = 0 = x0$ for all $x \in X$;
where δ is Kronecker Delta.

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Networks $\Rightarrow$ Right $*$-abundant semigroups

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(NR1) $\mathbf{s}(t)t = t = t\mathbf{r}(t)$ for $t \in T \cup T^0 \cup T^{-1}$;
(NR2) $t_1 t_2 = 0$ if $\mathbf{r}(t_1) \cap \mathbf{s}(t_2) = \emptyset$ for $t_1, t_2 \in T \cup T^0 \cup T^{-1}$;
(NR3) $t_1^{-1} t_2 = 0$ if $t_1 \neq t_2$ for $t_1, t_2 \in T$;
(NR4) $t^{-1} t = \mathbf{r}(t)$ for $t \in T$;
(NR5) $t^{-1} A = 0$ if $\mathbf{s}(t) \neq A$ for $t \in T$ and $A \in T^0$;
(NR6) $0x = 0 = x0$ for all $x \in X$.

Remark: for $A, B \in T^0$ we regard AB as a path from A to B if $A \cap B \neq \emptyset$.

For any path $\alpha = t_1 t_2 \cdots t_n \in P(\Gamma) \setminus \{0\}$, we define

$$\alpha^{-1} = {t_n}^{-1} \cdots {t_2}^{-1} {t_1}^{-1}.$$

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Networks ⇒ Right ∗-abundant semigroups

Proposition 1 The reduction system $(X^+, \to)$ where $X = T \cup T^0 \cup T^{-1}$,

$$u \to v \Leftrightarrow (u = xu_iy, v = xv_iy \text{ for some } x, y \in X^*, (u_i, v_i) \in R)$$

is a confluent rewriting system.

Outline of Proof: show the one-step case $(t_1t_2)t_3 = t_1(t_2t_3)$ for $t_1, t_2, t_3 \in T \cup T^0 \cup T^{-1}$, that is, consider the situation

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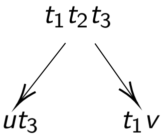

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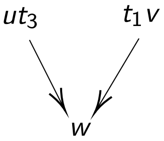

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(NR5) $t^{-1} A = 0$ if $\mathbf{s}(t) \neq A$ for $t \in T$ and $A \in T^0$;
(NR6) $0x = 0 = x0$ for all $x \in X$.

Remark: If we set $AB = A \cap B$ then for $t \in T$, $A \in T^0$ and $\mathbf{r}(t^{-1}) \subsetneq A$, then we have

$$t^{-1}\mathbf{r}(t^{-1})A = (t^{-1}\mathbf{r}(t^{-1}))A \to t^{-1}A \to 0$$

by (NR1) and (NR5), and also we have

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Networks $\Rightarrow$ Right $*$-abundant semigroups

(NR1) $\mathbf{s}(t)t = t = t\mathbf{r}(t)$ for $t \in T \cup T^0 \cup T^{-1}$;
(NR2) $t_1 t_2 = 0$ if $\mathbf{r}(t_1) \cap \mathbf{s}(t_2) = \emptyset$ for $t_1, t_2 \in T \cup T^0 \cup T^{-1}$;
(NR3) $t_1^{-1} t_2 = 0$ if $t_1 \neq t_2$ for $t_1, t_2 \in T$;
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Networks $\Rightarrow$ Right $*$-abundant semigroups

Certainly, the reduction system $(X^+, \to)$ where $X = T \cup T^0 \cup T^{-1}$ is a noetherian rewriting system. Consequently, every element of Q_Γ has a unique normal form as a word in X^+.

Definition 8 A path $\alpha = t_1 t_2 \dots t_n$ is **reduced** (or **irreducible**) if

$$t_{n-1} \neq \mathbf{s}(t_n), t_{i-1} \neq \mathbf{s}(t_i) \text{ and } t_{i+1} \neq \mathbf{r}(t_i)$$

for $1 < i < n$, where $t_1, t_2, \dots, t_n \in T \cup T^0$ and $n \in \mathbb{N}$.

Lemma 1 For any non-zero path $\alpha \in P(\Gamma)$, $\mathbf{s}(\alpha) = \mathbf{s}(\alpha')$ and $\mathbf{r}(\alpha) = \mathbf{r}(\alpha')$, where α' is the unique reduced path such that $[\alpha] = [\alpha']$.

We put

$$RP(\Gamma) = \{\alpha \in P(\Gamma) : \alpha \text{ is reduced}\}$$

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Networks ⇒ Right $*$-abundant semigroups

Theorem 1 Each element of Q_Γ has a unique normal form of one of the following types:

$$\text{(i) } [\alpha]; \ \text{(ii) } [\beta^{-1}]; \ \text{(iii) } [\alpha\beta^{-1}] \text{ and (iv) } [0],$$

where $\alpha \in RP(\Gamma)$, $\beta \in RLP(\Gamma)$ and in (iii) $\mathbf{r}(\alpha) \cap \mathbf{r}(\beta) \neq \emptyset$.

We will say that a word $w = \alpha\beta^{-1}$ with $\alpha \in RP(\Gamma)$, $\beta \in RLP(\Gamma)$ and $\mathbf{r}(\alpha) = \mathbf{r}(\beta)$ is a **right normal form**.

(i) $[\alpha] = [\alpha\beta^{-1}]$where $\beta = \mathbf{r}(\alpha)$;

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(iii)

$$[\alpha\beta^{-1}] = \begin{cases} [\alpha\beta^{-1}] & \text{if } \mathbf{r}(\alpha) = \mathbf{r}(\beta) \\ [\mu\beta^{-1}] & \text{otherwise,} \end{cases}$$

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Networks $\Rightarrow$ Right $*$-abundant semigroups

Corollary 1 Each non-zero element of Q_Γ has a unique right normal form representative $\alpha\beta^{-1}$, where $\alpha \in RP(\Gamma)$, $\beta \in RLP(\Gamma)$ and $\mathbf{r}(\alpha) = \mathbf{r}(\beta)$.

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Networks $\Rightarrow$ Right $*$-abundant semigroups

Lemma 2 For any $[\alpha\beta^{-1}], [\mu\nu^{-1}] \in Q_\Gamma$ with $\alpha\beta^{-1}$ and $\mu\nu^{-1}$ in right normal form, we have

$$[\alpha\beta^{-1}][\mu\nu^{-1}] = \begin{cases} [\alpha\mu\nu^{-1}] & \text{if } \beta = \mathbf{r}(\alpha) \text{ and } \mathbf{r}(\alpha) \cap \mathbf{s}(\mu) \neq \emptyset \\ [\alpha\xi\nu^{-1}] & \text{if } \beta \in RLP(\Gamma) \setminus T^0, \mu = \beta\xi \text{ for } \xi \in RP(\Gamma) \\ [\alpha(\nu\eta)^{-1}] & \text{if } \beta \in RLP(\Gamma) \setminus T^0, \beta = \mu\eta \text{ for } \eta \in RP(\Gamma) \\ [0] & \text{otherwise.} \end{cases}$$

Lemma 3 The idempotent set of Q_Γ is

$$E(Q_\Gamma) = \{[\alpha\alpha^{-1}] : \alpha \in RLP(\Gamma)\} \cup \{[0]\},$$

which contains a subsemilattice

$$E = E(Q_\Gamma) \setminus \{[A] : A \in T^0\} = \{[\alpha\alpha^{-1}] : \alpha \in RLP(\Gamma) \setminus T^0\} \cup \{0\}.$$

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Lemma 5 Suppose that $[\alpha\beta^{-1}]$, $[\mu\nu^{-1}] \in Q_\Gamma \setminus \{[0]\}$ where $\alpha\beta^{-1}$ and $\mu\nu^{-1}$ are in right normal form.

(i) $[\alpha\beta^{-1}] \ \mathcal{L}^* \ [\mu\nu^{-1}]$ if and only if $\beta = \nu$;

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(iii) if $\alpha \in RLP(\Gamma)$ and $\mu \in RP(\Gamma) \setminus RLP(\Gamma)$, then $([\alpha\beta^{-1}], [\mu\nu^{-1}]) \notin \mathcal{R}^*$.

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Right ample semigroups

Put
$S_\Gamma = \{[\alpha\beta^{-1}] \in Q_\Gamma : \alpha \in PR(\Gamma),\ \beta \in RLP(\Gamma) \setminus T^0,\ \mathbf{r}(\alpha) = \mathbf{r}(\beta)\} \cup \{[0]\}.$

$$[\alpha\beta^{-1}][\mu\nu^{-1}] = \begin{cases} [\alpha\xi\nu^{-1}] & \text{if } \mu = \beta\xi \text{ for some } \xi \in RP(\Gamma) \\ [\alpha(\nu\eta)^{-1}] & \text{if } \beta = \mu\eta \text{ for some } \eta \in RP(\Gamma) \\ [0] & \text{otherwise.} \end{cases}$$

Clearly, the subsemilattice

$$E = E(Q_\Gamma) \setminus \{[A] : A \in T^0\} = \{[\alpha\alpha^{-1}] : \alpha \in RLP(\Gamma) \setminus T^0\} \cup \{0\}$$

is the set of all idempotents of S_Γ.

For all $[\xi\xi^{-1}] \in E$ and $[\alpha\beta^{-1}] \in S_\Gamma \setminus \{[0]\}$,

$$[\xi\xi^{-1}][\alpha\beta^{-1}] = [\alpha\beta^{-1}]([\xi\xi^{-1}][\alpha\beta^{-1}])^*.$$

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Clearly, $R_\Gamma \subseteq S_\Gamma$ and

$$E(R_\Gamma) = E(S_\Gamma) = \{[\alpha\alpha^{-1}] : \alpha \in RLP(\Gamma) \setminus T^0\} \cup \{0\}.$$

Corollary 2 The semigroup R_Γ is a fundamental inverse subsemigroup of S_Γ.

Inverse semigroups

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Congruence-free

A semigroup is said to be **congruence-free** if it has only two congruences, the identity congruence and the universal congruence.

A right $*$-abundant semigroup is said to be $*$-**congruence-free** if it has only two unary semigroup congruences, the identity congruence and the universal congruence.

A $*$-**ideal** of a right abundant semigroup S is an ideal of S which is closed under the relation $\mathcal{L}^*$. It is easy to see that if we regard a right $*$-abundant semigroup S as a unary semigroup and I is a proper $*$-ideal of S then $\rho_I = (I \times I) \cup 1_S$ is a unary semigroup congruence on S.

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Congruence-free

Lemma 7 Let $I = \{[\alpha\beta^{-1}] \in Q_\Gamma : \alpha \in RP(\Gamma) \setminus RLP(\Gamma)\} \cup \{[0]\}$. Then I is a proper ideal of Q_Γ. Further $\rho_I = (I \times I) \cup 1_{Q_\Gamma}$ is an idempotent separating congruence on Q_Γ.

Lemma 8 Let $I = \{[\alpha\beta^{-1}] \in S_\Gamma : \alpha \in RP(\Gamma) \setminus RLP(\Gamma)\} \cup \{[0]\}$. Then I is a proper ideal of S_Γ. Further $\rho_I = (I \times I) \cup 1_{S_\Gamma}$ is an idempotent separating congruence on S_Γ.

Q_Γ and S_Γ are never congruence-free if they have non-regular elements. Notice that if t is a relation in Γ with $|\mathbf{s}(t)| > 1$ and $v \in \mathbf{s}(t)$, then we get $vt \in RP(\Gamma) \setminus RLP(\Gamma)$ and so $[vtt^{-1}] \in S_\Gamma \subseteq Q_\Gamma$ is a non-regular element.

Theorem 3 If $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ is a network and there exists $t \in T$ with $|\mathbf{s}(t)| > 1$, then Q_Γ and S_Γ are not congruence-free as a semigroup.

Congruence-free

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Q_Γ and S_Γ are never congruence-free if they have non-regular elements. Notice that if t is a relation in Γ with $|\mathbf{s}(t)| > 1$ and $v \in \mathbf{s}(t)$, then we get $vt \in RP(\Gamma) \setminus RLP(\Gamma)$ and so $[vtt^{-1}] \in S_\Gamma \subseteq Q_\Gamma$ is a non-regular element.

Theorem 3 If $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ is a network and there exists $t \in T$ with $|\mathbf{s}(t)| > 1$, then Q_Γ and S_Γ are not congruence-free as a semigroup.

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Q_Γ Congruence $*$-free

Let $\Gamma = (V, T, \mathbf{r}, \mathbf{s})$ be a network. For each $A \subset V$ the cardinality of the set $\{t \in T : \mathbf{s}(t) = A\}$ is called the *out-index* of A in Γ, denoted by $o(A)$.

Lemma 9 Let $\Gamma = (V, T, \mathbf{r}, \mathbf{s})$ be a network and let $t \in T$ be such that $o(\mathbf{r}(t)) = 0$ and there does not exist $A \in T^0 \setminus V$ with $\mathbf{r}(t) \subseteq A$. Then the principal ideal I generated by $[tt^{-1}]$ is a proper $*$-ideal of Q_Γ, where

$$\begin{aligned} I &= Q_\Gamma[tt^{-1}]Q_\Gamma \\ &= \{[\alpha\beta^{-1}] : \alpha \in RP(\Gamma), \beta \in RLP(\Gamma), \mathbf{r}(\alpha) = \mathbf{r}(\beta) = \mathbf{r}(t)\} \cup \{[0]\}. \end{aligned}$$

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S_Γ Congruence $*$-free

Lemma 10 Let $\Gamma = (V, T, \mathbf{r}, \mathbf{s})$ be a network with $|T| > 1$ and let $t, q \in T$ be such that $o(\mathbf{r}(t)) = 0$, $\mathbf{r}(t) \neq \mathbf{r}(q)$ and there does not exist $A \in T^0 \setminus V$ with $\mathbf{r}(t) \subseteq A$. Then the principal ideal generated by $[tt^{-1}]$ is a proper $*$-ideal of S_Γ, where

$$\begin{aligned} I &= S_\Gamma^1[tt^{-1}]S_\Gamma^1 \\ &= \{[\alpha\beta^{-1}] : \alpha \in RP(\Gamma), \beta \in RLP(\Gamma) \setminus T^0, \mathbf{r}(\alpha) = \mathbf{r}(\beta) = \mathbf{r}(t)\} \cup \{[0]\}. \end{aligned}$$

Theorem 5 If $\Gamma = (V, T, \mathbf{r}, \mathbf{s})$ is a network with $|T| > 1$ and let $t, q \in T$ be such that $o(\mathbf{r}(t)) = 0$, $\mathbf{r}(t) \neq \mathbf{r}(q)$ and there does not exist $A \in T^0 \setminus V$ with $\mathbf{r}(t) \subseteq A$, then S_Γ is not $*$-congruence-free as a unary semigroup.

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R_Γ Congruence-free

Lemma 11 Let $\Gamma = (V, T, \mathbf{r}, \mathbf{s})$ be a network with $|T| > 1$ and let $t, q \in T$ be such that $o(\mathbf{r}(t)) = 0$, $\mathbf{r}(t) \neq \mathbf{r}(q)$ and there does not exist $A \in T^0 \setminus V$ with $\mathbf{r}(t) \subseteq A$. Then the principal ideal I generated by $[tt^{-1}]$ is a proper ideal of R_Γ, where

$$\begin{aligned} I &= R_\Gamma[tt^{-1}]R_\Gamma \\ &= \{[\alpha\beta^{-1}] : \alpha, \beta \in RLP(\Gamma) \setminus T^0, \mathbf{r}(\alpha) = \mathbf{r}(\beta) = \mathbf{r}(t)\} \cup \{[0]\}. \end{aligned}$$

Theorem 6 If $\Gamma = (V, T, \mathbf{r}, \mathbf{s})$ is a network with $|T| > 1$ and let $t, q \in T$ be such that $o(\mathbf{r}(t)) = 0$, $\mathbf{r}(t) \neq \mathbf{r}(q)$ and there does not exist $A \in T^0 \setminus V$ with $\mathbf{r}(t) \subseteq A$, then R_Γ is not congruence-free.

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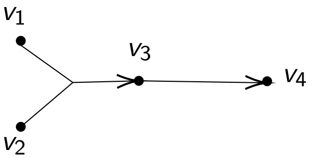
An example

Let $\Gamma = (V, T)$ be a network as shown in the following, where

$$V = \{v_1, v_2, v_3, v_4\} \text{ and } T = \{t_1, t_2\},$$

where $t_1 = (\{v_1, v_2\}, \{v_3\})$ and $t_2 = (\{v_3\}, \{v_4\})$.

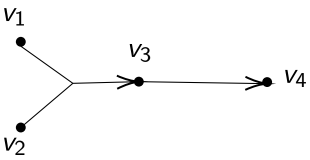


We get

$$T^0 = \{A\} \cup V,$$

where $A = \{v_1, v_2\}$.

An example



Let $X = \{v_1, v_2, A\}$, where $A = \{v_1, v_2\}$,

$X_{v_1,v_2} = \{w \in X^+ : v_1v_1,\ v_2v_2, AA, v_1v_2$ and v_2v_1 are not subwords in $w\}$

and

$$X_A = \{w \in X_{v_1,v_2} : w = \mu x, x \in X \setminus \{A\}\}.$$

We have

$$RP(\Gamma) = T \cup T^0 \cup X_{v_1,v_2} \cup X_A t_1 \cup X_A t_1 t_2 \cup \{t_1 t_2\}$$

and

$$RLP(\Gamma) = T \cup T^0 \cup \{t_1 t_2\}.$$

An example

$$
\begin{aligned}
Q_\Gamma = {} & \{[\alpha\mathbf{r}(\alpha)] : \alpha \in RP(\Gamma)\} \cup \{[\mathbf{r}(\beta)\beta^{-1}] : \beta \in RLP(\Gamma)\} \\
& \cup \{[\alpha\beta^{-1}] : \alpha, \beta \in RLP(\Gamma) \text{ and } \mathbf{r}(\alpha) = \mathbf{r}(\beta)\} \\
& \cup \{[\alpha t_1^{-1}] : \alpha \in X_A A \cup X_A t_1\} \\
& \cup \{[\alpha t_2^{-1}] : \alpha \in X_A t_1 t_2\} \cup \{[\alpha(t_1 t_2)^{-1}] : \alpha \in X_A t_1 t_2\} \cup \{[0]\},
\end{aligned}
$$

$$
\begin{aligned}
S_\Gamma = {} & \{[\mathbf{r}(\beta)\beta^{-1}] : \beta \in RLP(\Gamma) \setminus T^0\} \\
& \cup \{[\alpha\beta^{-1}] : \alpha \in RLP(\Gamma), \beta \in RLP(\Gamma) \setminus T^0 \text{ and } \mathbf{r}(\alpha) = \mathbf{r}(\beta)\} \\
& \cup \{[\alpha t_1^{-1}] : \alpha \in X_A A \cup X_A t_1\} \\
& \cup \{[\alpha t_2^{-1}] : \alpha \in X_A t_1 t_2\} \cup \{[\alpha(t_1 t_2)^{-1}] : \alpha \in X_A t_1 t_2\} \cup \{[0]\},
\end{aligned}
$$

and

$$
\begin{aligned}
R_\Gamma &= \{[\alpha\beta^{-1}] : \alpha, \beta \in RLP(\Gamma) \setminus T^0 \text{ and } \mathbf{r}(\alpha) = \mathbf{r}(\beta)\} \cup \{[0]\} \\
&= \{[t_1 t_1^{-1}],\ [t_2 t_2^{-1}],\ [t_1 t_2 t_2^{-1}],\ [t_1 t_2 (t_1 t_2)^{-1}],\ [0]\}.
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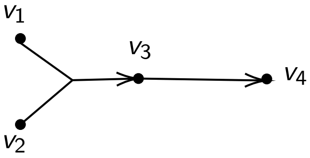
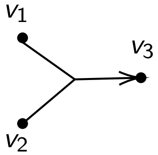
Lemma 9 Let $\Gamma = (V, T, \mathbf{r}, \mathbf{s})$ be a network and let $t \in T$ be such that $o(\mathbf{r}(t)) = 0$ and there does not exist $A \in T^0 \setminus V$ with $\mathbf{r}(t) \subseteq A$. Then the principal ideal I generated by $[tt^{-1}]$ is a proper $*$-ideal of Q_Γ.



t_1 :



$s(t_1) = \{v_1, \ v_2\}$

$r(t_1) = \{v_3\}$

$V = \{v_1, \ v_2, \ v_3, \ v_4\}$

$T = \{t_1, t_2\}$

t_2 :

$s(t_2) = \{v_3\}$

$r(t_2) = \{v_4\}$

We have

$$T^0 = \{A\} \cup V,$$

where $A = \{v_1, v_2\}$.

An example

Set

$$
\begin{aligned}
I_1 &= Q_\Gamma[t_2 t_2^{-1}]Q_\Gamma \\
&= \{[\alpha\beta^{-1}] : \alpha \in RP(\Gamma), \beta \in RLP(\Gamma), \mathbf{r}(\alpha) = \mathbf{r}(\beta) = \mathbf{r}(t_2)\} \cup \{[0]\} \\
&= \{[\alpha\mathbf{r}(t_2)] : \alpha \in \{t_2, \mathbf{r}(t_2), t_1t_2\} \cup X_A t_1 t_2\} \cup \{[\mathbf{r}(t_2)\beta^{-1}] : \beta \in \{t_2, t_1t_2\}\} \\
&\quad \cup \{[\alpha\beta^{-1}] : \alpha, \beta \in \{t_2, t_1t_2\}\} \\
&\quad \cup \{[\alpha t_2^{-1}] : \alpha \in X_A t_1 t_2\} \cup \{[\alpha(t_1t_2)^{-1}] : \alpha \in X_A t_1 t_2\} \cup \{[0]\},
\end{aligned}
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Then I_1 is a proper $*$-ideal of Q_Γ.

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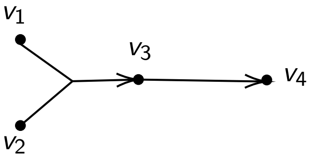

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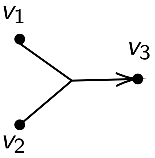

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t_2 :

$s(t_2) = \{v_3\}$

$r(t_2) = \{v_4\}$

We have $T^0 = \{A\} \cup V$ where $A = \{v_1, v_2\}$.

An example

Set

$$
\begin{aligned}
I_2 &= S_\Gamma^1[t_2 t_2^{-1}]S_\Gamma^1 \\
&= \{[\alpha\beta^{-1}] : \alpha \in RP(\Gamma), \beta \in RLP(\Gamma) \setminus T^0, \mathbf{r}(\alpha) = \mathbf{r}(\beta) = \mathbf{r}(t_2)\} \cup \{[0]\} \\
&= \{[\mathbf{r}(t_2)\beta^{-1}] : \beta \in \{t_2, t_1 t_2\}\} \\
&\qquad \cup \{[\alpha\beta^{-1}] : \alpha, \beta \in \{t_2, t_1 t_2\}\} \\
&\qquad \cup \{[\alpha t_2^{-1}] : \alpha \in X_A t_1 t_2\} \cup \{[\alpha(t_1 t_2)^{-1}] : \alpha \in X_A t_1 t_2\} \cup \{[0]\}
\end{aligned}
$$

Then I_2 is a proper $*$-ideal of S_Γ;

An example

Set

$$\begin{aligned} I_3 &= R_\Gamma[t_2 t_2^{-1}]R_\Gamma \\ &= \{[\alpha\beta^{-1}] : \alpha, \beta \in RLP(\Gamma) \setminus T^0, \mathbf{r}(\alpha) = \mathbf{r}(\beta) = \mathbf{r}(t_2)\} \cup \{[0]\} \\ &= \cup\{[\alpha\beta^{-1}] : \alpha, \beta \in \{t_2, t_1 t_2\}\} \cup \{[0]\}. \end{aligned}$$

Then I_3 is a proper ideal of R_Γ.

Homomorphisms

Definition 9 A **homomorphism** $\phi = (\phi_V, \phi_T)$ from $\Gamma = (V_\Gamma, T_\Gamma, \mathbf{s}, \mathbf{r})$ to $\Delta = (V_\Delta, T_\Delta, \mathbf{s}, \mathbf{r})$ consists of two maps $\phi_V : V_\Gamma \to V_\Delta$ and $\phi_T : T_\Gamma \to T_\Delta$ such that for all $t \in T_\Gamma$

$$\mathbf{s}(t)\phi = \{v\phi_V : v \in \mathbf{s}(t)\} = \mathbf{s}(t\phi_T)$$

and

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A mapping $\phi = (\phi_V, \phi_T)$ from $\Gamma = (V_\Gamma, T_\Gamma, \mathbf{s}, \mathbf{r})$ to $\Delta = (V_\Delta, T_\Delta, \mathbf{s}, \mathbf{r})$, is called **bijective** if ϕ_V and ϕ_T are bijective. A bijective homomorphism ϕ is an **isomorphism**.

Networks Γ and Δ are said to be **isomorphic** if there exists an isomorphism from Γ to Δ. If two networks Γ and Δ are isomorphic we write it as $\Gamma \cong \Delta$.

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Homomorphisms

Let S be an semigroup with a set $E(S)$ of all idempotents. The relation $\leq$ defined by for all $a, b \in S$,

$$a \leq b \text{ if and only if } a = xb = by,\ xa = a$$

for some $x, y \in S^1$, is a partial order on S called the **natural partial order** of S [5]. When the natural partial order is restricted to the set $E(S)$ it is as follows: for all $e, f \in E(S)$,

$$e \leq f \text{ if and only if } e = ef = fe.$$

Further, $E(S)$ is a partially ordered set with respect to $\leq$. In particular, if $E(S)$ is a semilattice, for all $e, f \in E$,

$$e \leq f \text{ if and only if } e = ef.$$

Homomorphisms

Lemma 12 Let $E(Q_\Gamma)$ be the set of all idempotents of Q_Γ and let $\leq$ be the natural partial order on Q_Γ defined above. Then the following statements hold.

(i) An idempotent $[\alpha\alpha^{-1}]$ is maximal in $E(Q_\Gamma)$ with respect to $\leq$ if and only if $\alpha \in T^0$;

(ii) An idempotent $[\alpha\alpha^{-1}]$ is maximal in $E = E(Q_\Gamma) \setminus \{[A] : A \in T^0\}$ with respect to $\leq$ if and only if $\alpha \in T$.

Theorem 7 Let $\Gamma = (V_\Gamma, T_\Gamma, \mathbf{s}, \mathbf{r})$ and $\Delta = (V_\Delta, T_\Delta, \mathbf{s}, \mathbf{r})$ be two networks. Then $\Gamma \cong \Delta$ if and only if $Q_\Gamma \cong Q_\Delta$.

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